Deciding tangles with weighted vertex sets and certifying large branch-width with tangle-kits

Jeffrey Kam

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1 Overview

Tangles are fundamental objects of study in the Graph Minors project. It serves as a notion of high connectivity components in a graph. However, the idea of a tangle is rather abstract, so it is natural to ask whether there exists a nice way to represent a tangle. Recently, Diestel [1] asked, given a graph $G = (V, E)$ and a tangle $\mathcal{T}$ of order k, whether there is a vertex subset $X \subset V$ that decides the tangle. In particular, for every separation (A, B) of order $< k$, we want to know if there is such an X where $(A, B) \in \mathcal{T}$ if and only if $|X \cap A| < |X \cap B|$.

In a recent paper, Elbracht et al. [3] proved a fractional version of this. They showed that we can decide a tangle through weighted vertex sets and we will be exploring the proof in this report. Additionally, we will also explore Oum and Seymour's paper [4] on certifying large branch-width with f-tangle-kits. At last, we will discuss some ideas that we could try to improve on the upper bound of the weight function.

2 Weighted Decider for Tangles

To define tangles, we have to first define separations. Different from our lectures, separations are defined in a slightly different way here, where we only consider vertex sets instead of the subgraphs.

Definition 2.1 Let $G = (V, E)$. A *separation* (A, B) is a pair of vertex subsets $A, B \subseteq V$ such that $A \cup B = V$ and there is no edge between $A \setminus B$ and $B \setminus A$.

The definition of tangle $\mathcal{T}$ of order k is defined exactly the same as we did in lecture, except the second axiom. Let $(A_1, B_1), (A_2, B_2), (A_3, B_3)$ be 3 separations of order $< k$. Then, instead of having $G \neq A_1 \cup A_2 \cup A_3$, we use the induced subgraph of the small sides, so $G \neq G[A_1] \cup G[A_2] \cup G[A_3]$.

Given a tangle $\mathcal{T}$ of order k, our goal is to first show that there exists a function $w : V \to \mathbb{R}^+$ that assigns weight to each vertex, such that for each separation (A, B) of order $< k$, if $w(A) < w(B)$, then $(A, B) \in \mathcal{T}$, where $w(X) = \sum_{v \in X} w(v)$ for $X \subseteq V$.

We need to first introduced a notion in the Graph Minors X paper. [5]

Definition 2.2 A separation $(A, B) \in \mathcal{T}$ is *extreme* if for all separation $(A', B') \in \mathcal{T}$ with $A' \subseteq A$ and $B \subseteq B'$, then $A = A'$ and $B = B'$.

Suppose we have a weight function $w : V \to \mathbb{R}^+$ that satisfies the constraints. Then, note that if we have 2 different separations (A, B) and (A', B') in the tangle with $A' \subseteq A$ and $B \subseteq B'$

$$w(A') \leq w(A) \leq w(B) \leq w(B')$$

Thus, it is natural to only consider the extreme separations of a tangle of order k.

In addition, we define a partial order for separations with respect to $\mathcal{T}$ that is related to extreme separations.

Definition 2.3 (Partial order of separations in a tangle)
Let $(A, B), (C, D)$ be two separations in a tangle $\mathcal{T}$. We define $(A, B) \leq (C, D)$ if and only if $A \subseteq C$ and $D \subseteq B$.

Note The separation (A, B) is extreme if and only if (A, B) is maximal with respect to the partial ordering. It is essentially pushing the small side to be as big as possible.

Before we get to the weighted decider proof, we need to prove a simple lemma.

Lemma 2.4 Let $\mathcal{T}$ be a tangle in a graph G of order k. If $(A, B), (C, D)$ are both extreme separations, then $|B \cap (C \cap D)| + |D \cap (A \cap B)| > |A \cap (C \cap D)| + |C \cap (A \cap B)|$.

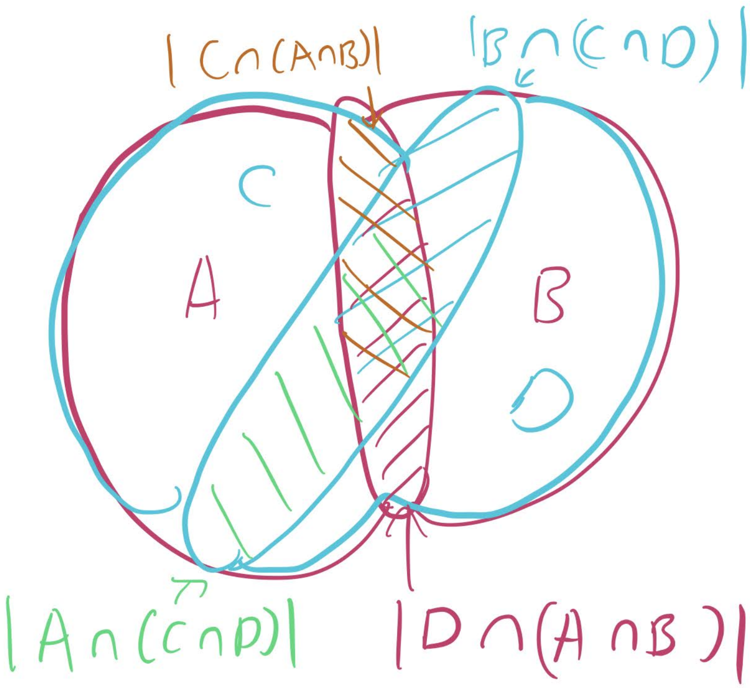

Figure 1: A rough illustration of lemma 2.4

Proof. Let (A, B) and (C, D) be two extreme separations of $\mathcal{T}$. Note that $(A \cup C, B \cap D)$ is also a separation under uncrossing, and that $(A, B) = (C, D) \leq (A \cup C, B \cap D)$. Since (A, B) and (C, D) are extreme, $(A \cup C, B \cap D)$ is not in the tangle $\mathcal{T}$. However, $(B \cap D, A \cup C)$ is also not in $\mathcal{T}$, because otherwise, we would have $A \cup C \cup (B \cap D)$ in the tangle, contradicting the second axiom. Thus, the order of $(A \cup C, B \cap D)$ is $\geq k$. Then, observe that

$$|A \cap B| + |C \cap D| = |(A \cup C) \cap (B \cap D)| + |(A \cap C) \cap (B \cup D)| \tag{1}$$

Since $|(A \cup C) \cap (B \cap D)| > k$, by the above equation, $|(A \cap C) \cap (B \cup D)| < k$, and so $|(A \cup C) \cap (B \cap D)| < |(A \cap C) \cap (B \cup D)|$. Then,

$$|(A \cup C) \cap (B \cap D)| < |(A \cap C) \cap (B \cup D)|$$
$$|(A \cup C) \cap (B \cap D)| + |(A \cap B) \cap (C \cap D)| < |(A \cap C) \cap (B \cup D)| + |(A \cap B) \cap (C \cap D)|$$
$$|A \cap (C \cap D)| + |C \cap (A \cap B)| < |B \cap (C \cap D)| + |D \cap (A \cap B)|$$

where the last step is due to equation 1. □

Note In lecture, we show that for a $\mathcal{T}$-independent set, then, for separations of small enough order ($< \frac{ord(\mathcal{T})}{2}$), we can decide which side is the big side, but the proof does not generalize.

This lemma shows that, in general, there are more vertices intersecting the boundary from the big side than from the small side, which is central to our proof. In addition, we also need the following linear programming result to prove the main theorem. For notational simplicity, we define $\mathbf{0}$ to be the 0 vector (of corresponding size), and that $x > \mathbf{0}$ if and only if $x_i > 0$ for each entry of x.

Theorem 2.5 (Tucker's Theorem [6])
Let $K \in \mathbb{R}^{n\times n}$ be a skew symmetric matrix. Then, there exists a real-valued vector $x \in \mathbb{R}^n$ such that

$$Kx \geq \mathbf{0} \text{ and } x \geq \mathbf{0} \text{ and } x + Kx > \mathbf{0}$$

Now, with lemma 2.4 and theorem 2.5, we can finally prove the main result below.

Theorem 2.6 Given a graph G and a tangle $\mathcal{T}$ of order k in G. There exists a function $w : V \to \mathbb{R}^+$ such that for any separation (A, B) of order $< k$, $(A, B) \in \mathcal{T}$ if and only if $w(A) < w(B)$.

Note w for vertex subset is defined the same way as before, namely $w(X) = \sum_{v\in X} w(v)$ for $X \subseteq V$.

Proof. Let $(A_1, B_1), \cdots, (A_m, B_m)$ be the extreme separations in $\mathcal{T}$. Then, we define a matrix $M \in \mathbb{R}^{n\times n}$ with entries

$$m_{i,j} = |B_i \cap (A_j \cap B_j)| - |A_i \cap (A_j \cap B_j)|.$$

Note that $m_{i,j} + m_{j,i} > 0$ since

$$|B_i \cap (A_j \cap B_j)| + |B_j \cap (A_i \cap B_i)| > |A_i \cap (A_j \cap B_j)| + |A_j \cap (A_i \cap B_i)|$$

by lemma 2.4. Thus, when we consider $M + M^T$, the matrix has all positive entries except at the diagonal, where they are all 0. Let

$$K' = \frac{M + M^T}{2} \text{ and } K = M - K'$$

then K is skew symmetric since

$$K^T = M^T - K'^T = \frac{M^T - M}{2} = -\frac{M - M^T}{2} = -K.$$

By theorem 2.5, we have a real valued vector $x > 0 \in \mathbb{R}^n$ such that $Kx \geq 0$ and $x + Kx > 0$. Using this, we define a weight function $w : V \to \mathbb{R}^+$ such that

$$w(v) = \sum_{i:v\in(A_i\cap B_i)} x_i.$$

Then, with some abuse of notation, for $X \subseteq V$, we define $w(X) = \sum_{v\in X} w(v)$.

We observe the following

$$
\begin{aligned}
(Mx)_i &= \sum_{j=1}^{n} x_j \cdot m_{ij} && (2)\\
&= \sum_{j=1}^{n} x_j \cdot (|B_i \cap (A_j \cap B_j)| - |A_i \cap (A_j \cap B_j)|) && (3)\\
&= \sum_{j=1}^{n} x_j \cdot |B_i \cap (A_j \cap B_j)| - \sum_{j=1}^{n} x_j \cdot |A_i \cap (A_j \cap B_j)| && (4)\\
&= \sum_{v \in B_i} w(v) - \sum_{v \in A_i} w(v) && (5)\\
&= w(B_i) - w(A_i) && (6)
\end{aligned}
$$

. So, $w(B_i) > w(A_i)$ if $(Mx)_i > 0$, and consequently, for w to be a valid weight function, all extreme separations (A_j, B_j) have to satisfy $w(B_j) > w(A_j)$. This implies it is sufficient to show $(Mx) > \mathbf{0}$.

From theorem 2.5, we know at least one entry of x is positive, since $x + Kx > 0$. We consider 2 following cases:

1. (≥ 2 entries in x are positive) Note that $M = K + K'$ and K' has positive entries except at the diagonal, and there are ≥ 2 positive entries in x, thus $K'x > 0$. By theorem 2.5, $Kx \geq 0$, so $M = (K + K')x > 0$.

2. (Exactly 1 entry in x is positive) Suppose x_i is the positive entry in x. For $j \neq i$, we have $(Mx)_j > 0$ since $K'x > 0$ and, by Tucker's theorem, $Kx \geq 0$. However, $(K'x)_i = 0$, which means $((K + K')x)_i$ could potentially be 0. To resolve this, we just pick a small enough $\epsilon > 0$ such that $w(A_j) + \epsilon < w(B_j)$ for all $j \not i$. Then, we can increase the weight of some vertex $v \in B_i \setminus A_i$ by ϵ, which would make $w(A_i) < w(B_i)$, while maintaining $w(A_j) < w(B_j)$ for all $j \neq i$. With this adjusted weight function, we then have $Mx > 0$.

Since both cases result in $Mx > 0$, we have the desired weight function w (adjusted in the second case) as required. □

Remark 2.7 The result extends directly to hypergraphs. Specifically, the proof above works for hypergraphs, since all the arguments made here only concern vertices. In addition, the weight function can have image in $\mathbb{N}$ since we can just approximate with positive rational numbers and multiply a large enough denominator to make it integral.

3 Weighted Decider for Edge-Tangles

Definition 3.1 Given a graph $G = (V, E)$. A *cut* is a bipartition (A, B) of V and the order is the number of edges with ends incident to both A and B. An *edge-tangle* $\mathcal{T}$ of order k consists of cuts of order $< k$ that satisfy the following properties:

1. Only one of (A, B) or (B, A) is in $\mathcal{T}$ for $|(A, B)| < k$.
2. For cuts (A_1, B_1), (A_2, B_2), (A_3, B_3) in $\mathcal{T}$, $B_1 \cap B_2 \cap B_3 \neq \emptyset$.
3. If $(A, B) \in \mathcal{T}$, then B is incident with $\geq k$ edges.

Here we briefly walk through the proof of an analogous theorem for edge-tangles. The proof provided by Elbracht et al. is more general than this statement, as it is proved for k-profile (see definition in the paper) and for graphs with weighted edges. [3] However, here we will only consider the unweighted case for simple graph, and use edge-tangle instead of k-profile as the idea is essentially the same.

First, we need some analogous definitions defined for edge-tangle.

Definition 3.2 Given a graph $G = (V, E)$, for each pair of vertices $(a, b) \in V^2$, we define $m : V^2 \to \mathbb{N}$, where $m(a, b)$ is the multiplicity of the edge ab. Then, we define the order in terms of m. Given a cut (A, B) of V, the *order* of (A, B) is $|(A, B)| = \sum_{(u,v) \in A \times B} m(u, v)$.

Note In a simple graph, $m(u, v) = 1$ if and only if there is an edge uv in E.

Definition 3.3 (Partial order of separations in an edge-tangle)
Let $(A, B), (C, D)$ be two cuts in an edge-tangle $\mathcal{T}$. We define the partial ordering the same way as definition 2.3, where $(A, B) \leq (C, D)$ if and only if $A \subseteq C$ and $D \subseteq B$. Again, we call a maximal cut with respect to this ordering an *extreme cut*.

With this, we have an analogous lemma stating that on average there are "more edges", represented as a pair of vertices here, intersecting the big side than the small side. The proof is similar to lemma 2.4.

Lemma 3.4 For every edge-tangle $\mathcal{T}$ and extreme cuts $(A, B), (C, D) \in \mathcal{T}$,

$$\sum_{(u,v) \in B^2 \cap (C \times D)} m(u, v) + \sum_{(u,v) \in D^2 \cap (A \times B)} m(u, v) >$$
$$\sum_{(u,v) \in A^2 \cap (C \times D)} m(u, v) + \sum_{(u,v) \in C^2 \cap (A \times B)} m(u, v)$$

With this lemma and Tucker's theorem, we can proceed to prove theorem 3.5. The proof structure is exactly the same as the proof of theorem 2.6 with everything rephrased in the analogous terms for edge-tangle. We outline the proof below.

Theorem 3.5 Given a graph G and an edge-tangle $\mathcal{T}$ of order k in G. There exists a function $w : V \to \mathbb{R}^+$ such that for any cuts (A, B) of order $< k$, $(A, B) \in \mathcal{T}$ if and only if $w(A) < w(B)$.

Proof (sketch).

1. Let $\mathcal{T}$ be an edge tangle of order k, $(A_1, B_1) \cdots (A_n, B_n)$ be the set of extreme cuts in $\mathcal{T}$.

2. We similarly define

$$m_{ij} = \sum_{(u,v)\in B_i^2\cap(A_j\times B_j)} m(u,v) - \sum_{(u,v)\in A_i^2\cap(A_j\times B_j)} m(u,v)$$

and the matrix $M = m_ij_{i,j\leq n}$ accordingly.

3. Like in theorem 2.6, we can find the vector x from Tucker's theorem. Then, we define our weight function

$$w'(u,v) = \sum_{i:(u,v)\in((A_i\times B_i)\cup(B_i\times A_i))} x_i \cdot m(u,v).$$

However, this is a weight function for pair of vertices, so we create an auxiliary weight function $w : V \to \mathbb{R}^+$ which fixes one of the vertex,

$$w(v) = \sum_{u\in V} w'(u,v)$$

4. Then, we evaluate $w(B_i) - w(A_i)$ like before, and with some computation, we have $w(B_i) - w(A_i) = 2(Mx)_i$ instead of only $(Mx)_i$. However, this doesn't matter since $w(B_i) - w(A_i) > 0$ if and only if $\frac{w(B_i)-w(A_i)}{2} > 0$.

5. Again, as in the proof of theorem 2.6, we have 2 cases, where there are 2 positive entries or exactly 1 positive entry in x. If we have 2, then we are done. Otherwise, we use the same ϵ-shifting trick to get our adjusted weight function, thus concluding our proof. $\square$

Remark 3.6 The proof above shows that it has a weighted decider for simple graphs and multi-graphs. The only difference is that $m(u,v)$ might potentially be greater than 1 for multi-graphs, but it does not change the proof. However, it does not extend to hypergraphs. The main reason is that the proof above view edges as a pair of vertices, which has a constant size 2, but for hypergraphs, that is not necessarily the case.

4 Counterexample for Edge-Tangle in Hypergraphs

We demonstrate below a concrete example when an edge-tangle $\mathcal{T}$ of a hypergraph has no weighted decider.

Example 4.1 Consider the hypergraph $H = (V,E)$, where V is the set of subsets of $[10] := \{1,\cdots,10\}$ of size 5, and E be the set of hyperedges defined as follow:

$$E = \{ \bigcup_{X\in V \text{ and } i\in X} \{X\} : i \in [10]\}.$$

Then, we have 10 hyperedges, each with $126 = \binom{9}{4}$ 5-element subsets. We want to show that H has an edge-tangle of order 10 with no weighted decider.

Proof. Let S be the set of all cuts of order < 10. Let (A, B) be a cut, then $(A, B) \in S$ if and only if not all 10 hyperedges have ends in both A and B, otherwise, (A, B) would have order 10. We define $\cup A := \bigcup_{v \in A} v$ and similarly for $\cup B$. Note that at least one of $\cup A$ or $\cup B$ has to contain all of [10]. Then,it is easy to show $(A, B) \in S$ if and only if one of $\cup A$ or $\cup B$ is a proper subset of [10]. (From our definition of the hyperedges, parameterized by some element $i \in [10]$, a hyperedge has an end in A if and only if some subsets of A contain i, which implies $i \in \cup A$).

Then, we define our edge-tangle $\mathcal{T}$ with this property so that the proper subset side is the "small" side, namely

$$\mathcal{T} := \{(A, B) \in S : \cup A \subsetneq [10]\}.$$

It is easy to verify that $\mathcal{T}$ is an edge-tangle. For details, please refer to the original paper. [3] Now, we proceed to prove that $\mathcal{T}$ has no weighted decider. Suppose, for a contradiction, $\mathcal{T}$ has a weighted decider $w : V \to \mathbb{R}^+$. Let (A_i, B_i) be cut of V where $A_i := \{S \in V : i \notin S\}$ and $B_i = \{S \in V : i \in S\}$, for all $i \in [10]$. Note that $(A_i, B_i) \in \mathcal{T}$ since $\cup A_i$ is a proper subset of [10]. Since w is a weighted decider for $\mathcal{T}$, we have $w(B_i) > w(A_i)$ for all $i \in [10]$. Observe that

$$\begin{aligned}
\sum_{i \in [k]} (w(B_i) - w(A_i)) &= \sum_{i \in [k]} (\sum_{v \in B_i} w(v) - \sum_{v \in A_i} w(v)) \\
&= \sum_{i \in [k]} (\sum_{v \in B_i} w(v) - \sum_{v \in A_i} w(v)) \\
&= \sum_{i \in [k]} (\sum_{v \in V : i \in v} w(v) - \sum_{v \in V : i \notin v} w(v) \\
&= \sum_{i \in [k]} (\sum_{v \in V : i \in v} w(v)) - \sum_{i \in [k]} (\sum_{v \in V : i \notin v} w(v)) \\
&= \sum_{v \in V} w(v)(|\{i \in [10] : i \in v\}|) - \sum_{v \in V} w(v)(|\{i \in [10] : i \notin v\}|) \\
&= \sum_{v \in V} w(v)(|\{i \in [10] : i \in v\}| - |\{i \in [10] : i \notin v\}|)
\end{aligned}$$

, and further note that,

$$\sum_{v \in V} w(v)(|\{i \in [10] : i \in v\}| - |\{i \in [10] : i \notin v\}|) = \sum_{v \in V} w(v)(5 - (10 - 5)) = 0.$$

However, since w is assumed to be a weighted decider,

$$\sum_{i \in [k]} (w(B_i) - w(A_i)) > 0.$$

This is a contradiction. Thus, the edge-tangle $\mathcal{T}$ of H does not have any weighted decider as required. $\square$

Note An analogous counterexample for matroids was independently found by Jim Geelen.

5 Branch-width and Tangle-kits

Definition 5.1 Let V be a finite set and $f : 2^V \to \mathbb{Z}$. f is *symmetric submodular* if and only if $f(X)+f(Y) > f(X\cap Y)+f(X\cup Y)$ and $f(X) = f(V\setminus X)$, for $X,Y \subseteq Y$.

Example 5.2 The connectivity function for a tangle matroid, namely

$$\lambda(X) := \min(|(H_1, H_2)| : (H_1, H_2) \in \mathcal{T} \text{ and } X \subseteq V(H_1))$$

is submodular, where $|(H_1, H_2)|$ is the order of (H_1, H_2), and $\mathcal{T}$ some tangle of order k for a graph $G = (V, E)$. However, it is not symmetric, since $\lambda(\emptyset) = 0 \neq ord(\mathcal{T}) = k = \lambda(V)$.

Now, if we define $P(X) = \lambda(X)+\lambda(V\setminus X)$, then P is symmetric and submodular. Let $X, Y \subseteq V$.

- Symmetric: $P(X) = \lambda(X) + \lambda(V \setminus X) = P(V \setminus X)$.
- Submodular:

$$\begin{aligned}
P(X) + P(Y) =& \lambda(X) + \lambda(Y) + \lambda(V \setminus X) + \lambda(V \setminus Y)\\
\geq& \lambda(X \cup Y) + \lambda(X \cap Y)\\
&+ \lambda((V \setminus X) \cup (V \setminus Y)) + \lambda((V \setminus X) \cap (V \setminus Y))\\
=& \lambda(X \cup Y) + \lambda(V \setminus (X \cup Y)) + \lambda(X \cap Y) + \lambda(V \setminus (X \cap Y))\\
=& P(X \cup Y) + P(X \cap Y)
\end{aligned}$$

Definition 5.3 Let f be a symmetric submodular function. We say f is a *connectivity function* if $f(\emptyset) = 0$ and $f(\{v\}) \leq 1$ for all $v \in V$.

In our lectures, we have shown how a branch-decomposition of a graph relates to a tangle. In particular, we showed the following.

Theorem 5.4 Given a graph G and a fixed $k \in \mathbb{N}$, we can effectively find either

- A branch-decomposition of width $\leq k$, or
- A tangle of order $\lceil \frac{k}{3} \rceil$

In fact, we can extend the idea of branch-width to matroids, and more generally, symmetric submodular functions.

Definition 5.5 Given a branch-decomposition (T, L) of a symmetric submodular function f, where T is a cubic tree and L a function mapping V to leaves of T. The width of a branch-decomposition is defined as

$$\max_{(X,Y) \text{ induced by } (T\setminus e) \;:\; e\in E[T]} f(L^{-1}(X))$$

where $E[T]$ are the edges of T. The *branch-width* of f is the minimum width over all possible branch-decompositions of f.

So, the natural question to ask is whether we have a similar theorem for general symmetric submodular function. Oum and Seymour [4] showed that if the function is a connectivity function, then we could find show that the connectivity function has branch-width $\geq k$ with a polynomial size certificate that can be checked in polynomial time (polynomial with respect to $|V|$).

Definition 5.6 Let $f : 2^V \to \mathbb{Z}$ be a connectivity function. We define the *f-tangle* $\mathcal{T}$ of order k to be a set of subsets of V satisfying the following 4 axioms:

1. For $A \in \mathcal{T}$, $f(A) < k$.
2. For all $A \subseteq V$, if $f(A) < k$, then only one of A or $V \setminus A$ is in $\mathcal{T}$.
3. If $A, B, C \in \mathcal{T}$, then $A \cup B \cup C \neq V$.
4. For all $v \in V$, $V \setminus \{v\} \notin \mathcal{T}$.

Definition 5.7 Let f be a connectivity function. For disjoint subsets $X, Y \subseteq V$, we define

$$f_{min}(X, Y) = \min_{X \subseteq U \subseteq (V \setminus Y)} f(U)$$

We want to define a new notion called a f-tangle-kit that is similar to a f-tangle. In particular, we want to show we have a f-tangle-kits of order k if and only if we have a f-tangle of order k. Recall from lecture again, Robertson and Seymour [5] proved that the branch-width is equal to the maximum order of tangle. So instead of dealing with a f-tangle, we can provide a f-tangle-kit of order $\geq k+1$ as a certificate of large branch-width.

Definition 5.8 Let f be a connectivity function on subsets of some finite set V. We define the *f-tangle-kits* of order k to be a pair (P, μ), where

$$P = \{(X, Y) : X, Y \subseteq V, X \cap Y = \emptyset, f_{min}(X, Y) = |X| = |Y| < k\}$$

and

$$\mu : P \to 2^V$$

that satisfies the following 3 axioms:

1. $\mu(X_1, Y_1) \cup \mu(X_2, Y_2) \cup \mu(X_3, Y_3) \neq V$ for $(X_1, Y_1), (X_2, Y_2), (X_3, Y_3) \in P$.
2. for all $(A, B) \in P$, there is no Z where $A \subseteq Z \subseteq V \setminus B$, $f(Z) = |A|$, $Z \nsubseteq \mu(A, B)$, and $V \setminus Z \nsubseteq \mu(B, A)$.
3. $|\mu(X, Y)| \neq |V| - 1$ for all $(X, Y) \in P$.

Note There is an alternative formulation for axiom 2 of f-tangle-kit: For all $x \in V \setminus (\mu(A, B) \cup B)$ and $y \in V \setminus (\mu(B, A) \cup A)$, if $x \neq y$, then $f_{min}(A \cup \{x\}, B \cup \{y\}) > |A|$).

Theorem 5.9 Let $f : 2^V \to \mathbb{Z}$ be a connectivity function. There exists a f-tangle of order k if and only if there exists a f-tangle-kit of order k.

Before we can prove the main theorem of this section (theorem 5.9), we have to first introduce the following lemma.

Lemma 5.10 Let $f : 2^V \to \mathbb{Z}$ be a connectivity function. Given a $Z \subseteq V$, there exists $X \subseteq Z$ and $Y \subseteq V \setminus Z$ such that $f_{min}(X, Y) = f(Z)$ and $|X| = |Y| = f(Z)$.

Proof (Sketch).
Choose $X \subseteq Z$ to be maximal where $f_{min}(X, V \setminus Z)$. Then, for $v \in Z \setminus X$, we have $|X| \leq f_{min}(X \cup \{v\}, V \setminus Z) \leq |X| + 1$ (by lemma 3.2 in [4]), and since X is chosen as maximal, $f_{min}(X \cup \{v\}, V \setminus Z) = |X|$, instead of $|X| + 1$. Consequently,

$$f_{min}(Z, V \setminus Z) = f_{min}(X \cup \{v : v \in Z \setminus X\}), Z) = f_{min}(X, Y) = |X|.$$

Similarly, we choose $Y \subseteq V \setminus Z$ to be maximal. By similar argument, we have

$$f_{min}(Z, V \setminus Z) = f_{min}(Z, Y \cup \{v : v \in (V \setminus Z) \setminus Y\}) = f_{min}(X, Y) = |Y|.$$

Since $f_{min}(Z, V \setminus Z) = \min_{Z \subseteq U \subseteq Z} f(U) = f(Z)$, so $f_{min}(X, Y) = f(Z) = |X| = |Y|$ as required. $\square$

Remark 5.11 When f is just a general symmetric submodular function, but not a connectivity function, the induction step of lemma 3.2 of Oum and Seymour's paper [4], on which our lemma 5.10 depends, will no longer be valid.

Proof (For theorem 5.9).
Suppose we have a f-tangle of order k. We want to show that there is a f-tangle-kit (P, μ) of order k. Let $P = \{(X, Y) : X, Y \subseteq V, X \cap Y = \emptyset, f_{min}(X, Y) = |X| = |Y| < k\}$

Claim For $(X, Y) \in P$, there exists a unique maximal set $Z \in \mathcal{T}$ such that $X \subseteq Z \subseteq V \setminus Y$ and $f(Z) = f_{min}(X, Y)$

Proof. Let $Z_1, Z_2 \in \mathcal{T}$ satisfying $X \subseteq Z_1 \subseteq V \setminus Y$, $X \subseteq Z_2 \subseteq V \setminus Y$, and $f(Z_1) = f(Z_2) = f_{min}(X, Y)$. Since $X \subseteq Z_1 \cap Z_2$ and $X \subseteq Z_1 \cup Z_2$, we have $f(Z_1 \cap Z_2)$ and $f(Z_1 \cup Z_2)$ both $\geq f_{min}(X, Y)$. Note that $2 f_{min} f(X, Y) = f(Z_1) + f(Z_2) \geq f(Z_1 \cap Z_2) + f(Z_1 \cup Z_2)$, so $f(Z_1 \cap Z_2) = f(Z_1 \cup Z_2) = f_{min}(X, Y)$. Also, $Z_1 \cup Z_2 \in \mathcal{T}$ since $V \setminus (Z_1 \cup Z_2)$ would have violated f-tangle axiom 3. Then, we can just take the union of all such Z_i satisfying the above and get our unique maximal subset Z. $\square$

With the claim, we define $\mu(X, Y) = Z$ and μ is well defined since Z is unique w.r.t. (X, Y). Then, we simply have to check that this (P, μ) satisfies all the f-tangle-kit axioms.

1. Tangle-kit axiom 1 is satisfied since $\mu(X,Y) \in \mathcal{T}$ and, for $Z_1, Z_2, Z_3 \in \mathcal{T}$, $Z_1 \cup Z_2 \cup Z_3 \neq V$ by tangle axiom 3, so $\mu(X_1,Y_1) \cup \mu(X_2,Y_2) \cup \mu(X_3,Y_3) \neq V$.

2. Tangle-kit axiom 2 is satisfied by construction of μ and tangle axiom 2. By tangle axion 2, either Z or $V \setminus Z$ is in the tangle. Suppose $Z \in \mathcal{T}$, then $Z \subseteq \mu(A,B)$ since $\mu(A,B)$ is constructed to be maximal, thus a contradiction.

3. Tangle-kit axiom 3 follows directly from tangle axiom 4.

Conversely, suppose we have a f-tangle-kit (P,μ) of order k. We want to construct a f-tangle of order k from the f-tangle-kit as follows. For $Z \subseteq V$ with $f(Z) < k$, we choose $(A,B) \in P$ such that $|X| = |Y| = f(Z)$ and $X \subseteq Z \subseteq V \setminus Y$. We know this exists because of lemma 5.10. Then, if $Z \subseteq \mu(A,B)$, $Z \in \mathcal{T}$; otherwise, $V \setminus Z \in \mathcal{T}$. This is indeed well-defined, but we won't verify here (see [4] for details). We have to then verify that this satisfies all the f-tangle axioms.

1. Tangle axiom 1 is satisfied by construction.

2. Similarly, since the construction is well-defined, tangle axiom 2 is automatically satisfied.

3. For tangle axiom 3, assume $A_1, A_2, A_3 \in \mathcal{T}$, then by construction, each A_1, A_2, A_3 are subsets of some $\mu(A_1,B_1), \mu(A_2,B_2), \mu(A_3,B_3)$. Thus, $A_1 \cup A_2 \cup A_3 \subseteq \mu(A_1, B_1 \cup \mu(A_2,B_2) \cup \mu(A_3,B_3) \neq V$ by tangle-kit axiom 1 as required.

4. For tangle axiom 4, suppose for a contradiction that $V \setminus \{v\} \in \mathcal{T}$ for some $v \in V$, then, there is some $(X,Y) \in P$ of the f-tangle-kit such that $V \setminus \{v\} \subseteq \mu(X,Y)$. This means $\mu(X,Y) = V$ or $\mu(X,Y) = V \setminus \{v\}$, which contradicts tangle-kit axiom 1 and 3 respectively.

Thus, we have shown there is a f-tangle-kit of order k if and only if there exists a f-tangle of order k as required. □

Theorem 5.12 Let $f : 2^V \to \mathbb{Z}$ be a connectivity function with branch-width $> k$, for some fixed k. Then, we can find a polynomial-sized certificate that it has branch-width $> k$ that can be verified in polynomial time, with respect to $|V|$.

Proof. Since a f-tangle of order k exists if and only if a f-tangle-kit of order k exists, by theorem 5.9, it is sufficient to provide a f-tangle-kit of order $\geq k+1$ as a certificate that f has branch-width $> k$. Now, suppose we are given (P,μ) as a certificate, we need to show it is of polynomial size, and can be verified in polynomial time.

Note that $|P| \leq \sum_{i=1}^{k} \binom{|V|}{i}^2$ since each pair of disjoint set $(A,B) \in P$ must have $|A| = |B|$ by definition. Since k is fixed, the certificate size is polynomial in $|V|$ as required.

Now, we need to show it can be checked in polynomial time. It is same as checking whether the certificate (P,μ) actually satisfy the f-tangle-kit axioms. It is known that we can calculate f_{min} in polynomial time in $|V|$ with submodular function minimization. Verifying the axioms 1 and 3 is simply polynomial, and for axiom 2, we consider its alternative formulation (see note 6), and then use the fact that f_{min} can be computed in polynomial time. $\square$

Remark 5.13 Given a connectivity function f and a fixed k, it is in $\mathsf{NP} \cap \mathsf{coNP}$ to decide whether f has branch-width at most k. The above theorem shows that this decision problem is in coNP, and since we can simply verify a branch-decomposition in polynomial time, if given one, this is also in NP.

6 Some Ideas for Improving Upper Bound

Here we suggest some ideas that might help improve the weight function, as the upper-bound is not polynomially bounded by the number of vertices. Below are some ideas or questions that we might be able further explore.

1. What about graphs under some restrictions, or simply different graph classes like planar graphs? Can we find a $\{0, 1\}$-decider for those graphs? For instance, Elbracht et al. showed that there exists a $\{0, 1\}$-decider for k-connected graphs with $\geq 4(k)$ vertices. [2]

2. Maybe only a subset of the extreme separations can already capture the essence that big sides intersect more with the boundary. For instance, in section 5, we only need a number, that is polynomial in $|V|$, of disjoint sets (X, Y) in the f-tangle-kits, whereas for f-tangles, this might be exponential in $|V|$.

3. Can we have some polynomial upper bound of the number of useful extreme separations? It seems to me that all the boundaries of extreme separations intersect each other, so maybe we can do something with the intersection of those boundaries, without considering each of them specifically.

4. For a specific graph class, like planar, maybe we can consider contracting to a grid, and each vertex stores some weight with respect to the number of contractions for that vertex.

5. Can we (and how to) view this problem about tangle in its dual notion, using branch-decomposition and its branch-width?

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